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Counterexamples in AB percolation

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Abstract. Halley conjectured that the AB percolation probability function attains its maximum at $p=\frac{1}{2}$, and that consequently infinite AB percolation is impossible on a bipartite graph whose standard site percolation critical probability is greater than $\frac{1}{2}$. We produce counterexamples that demonstrate that both conjectures are false. In fact, for any positive integer N, a graph may be constructed which has at least N AB percolation transitions.

1. Introduction

Let the vertices of an infinite lattice graph G be independently labelled A with probability p and B with probability $1-p$. Connect adjacent vertices of G which have opposite labels with a bond, while adjacent vertices with the same labels are not connected. This variant of the percolation model was introduced by Mai and Halley (1980), as 'AB percolation', for the study of gelation and polymerisation processes, and independently by Sevsek *et al* (1983) as 'antipercolation', for the study of antiferromagnetism.

An edge of G is an *AB bond* if the endpoints have different labels. A path is an *AB path* if all its edges are AB bonds. The *AB cluster* containing a vertex v, denoted W_v^{AB}, is the set of all vertices that may be reached from v through an AB path. The number of vertices in W_v^{AB} is denoted by $|W_v^{AB}|$. Define the *AB percolation probability* by

$$\theta_v^{AB}(p, G) = P_p[|W_v^{AB}| = +\infty]$$

where P_x denotes the probability measure when the parameter value is x. Halley (1983) showed that the AB percolation probability is symmetric about $\frac{1}{2}$ for any graph G:

$$\theta_v^{AB}(p, G) = \theta_v^{AB}(1-p, G)$$

for all $p \in [0, 1]$ and all vertices $v \in G$. If G is connected, the set of values $p \in [0, 1]$ for which $\theta_v^{AB}(p, G) > 0$ is independent of the choice of v. A value of p which separates intervals where $\theta_v^{AB} > 0$ and $\theta_v^{AB} = 0$ is an *AB percolation threshold.*

Halley (1983) considered AB percolation on bipartite graphs. A graph G is *bipartite* if there exists a partition of the vertex set into two sets V_1 and V_2 such that every edge of G has one endpoint in V_1 and one endpoint in V_2. An alternate characterisation is that G is bipartite if and only if every cycle in G has an even number of edges. Halley showed that, if G is a bipartite graph with standard site percolation critical probability greater than $\frac{1}{2}$, then AB percolation does not occur when $p=\frac{1}{2}$: $\theta_v^{AB}(\frac{1}{2}, G) = 0$. He conjectured that the AB percolation probability attains its maximum value when

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$p=\frac{1}{2}$. If true, this would imply a second conjecture: that infinite AB percolation clusters are impossible for all $p \in [0, 1]$ on a bipartite graph G which has site percolation critical probability greater than $\frac{1}{2}$. If the second conjecture were true, the AB percolation existence problem for bipartite graphs would be nearly settled, since Wierman (1988a) showed that AB percolation exists on any graph with standard site percolation critical probability strictly less than $\frac{1}{2}$.

Previous results are consistent with Halley's conjectures. Appel and Wierman (1987) gave the first rigorous proof that infinite AB percolation does not occur for any parameter value in a certain class of bipartite planar graphs (including the square lattice). If G is a bipartite graph with bipartition sets V_1 and V_2, construct a graph H_i from each V_i by joining u and v in V_i if and only if u and v have a common neighbour in G. Denoting the classical site percolation critical probability of a graph H by $p_c(H)$, Appel and Wierman showed that AB percolation is impossible on a bipartite graph G if $p_c(H_1)+p_c(H_2)>1$, and also if this sum equals one under certain symmetry and periodicity conditions on G. For example, their results show that AB percolation is impossible on the hexagonal and square lattices. Wierman (1988b) showed that AB percolation is impossible on a bipartite graph G if $p_c(H_1)+p_c(H_2)=1$, without additional conditions. Furthermore, he exhibited some bipartite graphs for which AB percolation is impossible in an interval containing $\frac{1}{2}$, but for which AB percolation was not ruled out at all other values of p.

In § 2, we provide a simple construction which produces bipartite graphs with standard percolation critical probability strictly greater than $\frac{1}{2}$, for which we show that AB percolation occurs at a value $p_0 \neq \frac{1}{2}$. These graphs are counterexamples to both of Halley's conjectures. The graphs constructed for the counterexamples are artificial or 'non-physical'. We know of no lattice in which all sites are indistinguishable for which this behaviour occurs.

Note that the probability that an edge in G is an AB bond, as well as the expected proportion of AB bonds in G, is equal to $p(1-p)$, which is maximised at $p=\frac{1}{2}$ and monotonic on each side of $\frac{1}{2}$. Thus, our examples show that the probability of an AB infinite cluster is not a monotonic function of either of these quantities.

A surprising result is proved in § 3. Given any finite set of points $\{p_i\}$ in$(\frac{1}{2}, 1)$ and constants $\delta_i>0$, a graph may be constructed for which AB percolation occurs at each p_i and does not occur at p unless $p_i-\delta_i<p<p_i+\delta_i$ for some i. A consequence of this result is that, for any positive integer N, there exists a graph which has at least N AB percolation thresholds.

These examples how that the behaviour of AB percolation models can be quite different from that of ordinary site or bond percolation models. The existence of multiple AB percolation thresholds also raises questions concerning equality or inequality of various critical exponents at all the thresholds.

2. A counterexample

We begin with a simple counterexample to Halley's conjectures, which illustrates aspects of our construction and proof technique in § 3.

We construct the graph $G(k)$ from the square lattice by replacing each edge by a set of $2k$ paths of two edges connecting the two endpoints of the original edge in the square lattice (see figure 1). For each $k \geqslant 1$, $G(k)$ is bipartite, with the inserted vertices forming one bipartition set and the original square lattice vertices forming the other.

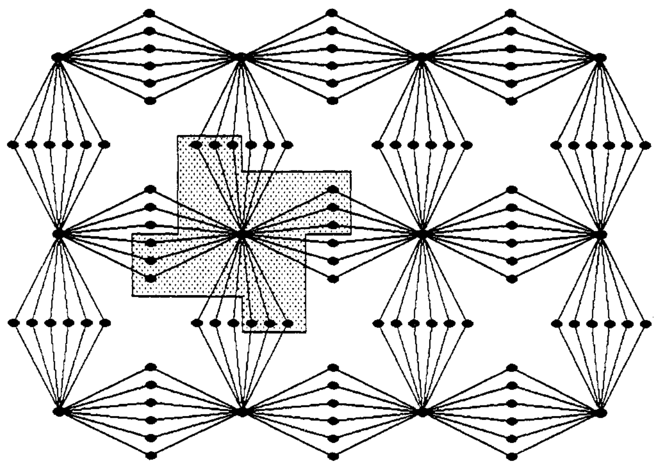

Figure 1. A portion of the graph $G(3)$, formed by replacing each edge of the square lattice by six paths of length two. The shaded region contains the twelve inserted vertices which are associated with the square lattice site in the centre of the region.

In the standard site percolation model on $G(k)$, any infinite open path must contain an infinite open path in the original square lattice. Since the site percolation critical probability of the square lattice is strictly greater than $\frac{1}{2}$, we have $p_c(G(k)) > \frac{1}{2}$ for all k.

Thus, $G(k)$ satisfies the hypotheses of Halley's conjectures. By Halley's (1983) result, there is no AB percolation on $G(k)$ when $p = \frac{1}{2}$. Since the critical probability of the graph constructed from one bipartition set has critical probability strictly greater than $\frac{1}{2}$, the result of Wierman (1988b) shows that AB percolation does not occur on an interval containing $\frac{1}{2}$. We will now prove that AB percolation does exist on $G(k)$ if $k \geq 15$.

Of the $2k$ vertices inserted to replace each edge in the square lattice to obtain $G(k)$, associate k with each of the endpoint vertices. For an AB percolation model on $G(k)$, define a standard percolation model on the square lattice by declaring each vertex to be open if it is labelled A and at least one of the associated vertices for each of its four edges is open. An infinite AB cluster exists on $G(k)$ if there is an infinite open cluster in the square lattice site percolation model, which occurs if $p[1-p^k]^4 \geq 0.7071 > p_c(S)$. The function $p[1-p^k]^4$ is maximised at $p = (1+4k)^{-1/k}$, with a maximum value of $(1+4k)^{-1/k}[4k/(1+4k)]^4$. Since the maximum value is greater than 0.7071 for $k \geq 15$, AB percolation exists on $G(k)$ for $k \geq 15$.

This construction can be applied with any graph G substituted for the square lattice. For any $\varepsilon > 0$, by choosing the original graph to have site percolation critical probability greater than $1-\varepsilon$, it produces a graph with AB percolation at a point in $(0, \varepsilon)$ and a point in $(1-\varepsilon, 1)$, but no AB percolation in $[\varepsilon, 1-\varepsilon]$. Thus there exist graphs on which AB percolation exists, but the length of the central interval of no AB percolation is as close to one as desired.

3. Existence of multiple thresholds

Our main result shows that there exists a graph with any specified number of AB percolation phase transitions, with the transitions occurring in specified intervals.

The key step in the proof is the construction of graph structures which essentially are traversed by AB paths only when p is near a selected value. Let $S=(F, u, v)$ be a triple, where F is a connected graph and u and v are distinguished vertices of F. Let $H(S)$ $\{H_B(S)$ respectively$\}$ denote the event that there exists an AB path from a neighbour of u to a neighbour of v {with those particular neighbours of u and v labelled B} when p is the parameter of the model. For $p \in [\frac{1}{2}, 1]$, $\delta > 0$, and $\varepsilon > 0$, S will be called (p, δ, ε) *selector* if

$$P_p[H_B(S)] > 1-\varepsilon$$

and for all $x \geq \frac{1}{2}$ such that $|x-p| > \delta$,

$$P_x[H(S)] < \varepsilon.$$

Lemma. There exists a (p, δ, ε) selector for every $p \in (\frac{1}{2}, 1)$, $\delta > 0$ and $\varepsilon > 0$.

Proof. We wish to construct a (p, δ, ε) selector for fixed $p \in (\frac{1}{2}, 1)$, $\delta > 0$ and $\varepsilon > 0$. For convenience, we give the argument for a rational number p. Then there exist relatively prime integers i and j such that $p = j/(i+j)$, with $j > i$ since $p > \frac{1}{2}$.

We now describe a family of graphs which provide selectors for p. The parameters l, m and n that appear will be chosen later. The basic unit of the selector is constructed from a path $v_0, v_1, v_2, \ldots, v_{2nj+1}$ of length $2nj+1$, by replacing each of the first $n(j-i)$ vertices with odd indices by m vertices, each connected to the previous and next vertices in the sequence. The selector $S_{l,m,n} = (G_{l,m,n}, a, b)$ consists of l copies of this basic unit connected in parallel, with common initial vertex a and common final vertex b (see figure 2).

Denote the probability that there exists an AB path connecting vertices adjacent to a and b in a particular basic unit by

$$g_{n,m}(p) \equiv p^{nj}(1-p)^{ni}[1(1-p)^m]^{n(j-i)} + (1-p)^{nj}p^{ni}[1-p^m]^{n(j-i)}.$$

Letting $h_l(x) = 1-(1-x)^l$, we have

$$P_p[H(S_{l,m,n})] = h_l(g_{n,m})$$

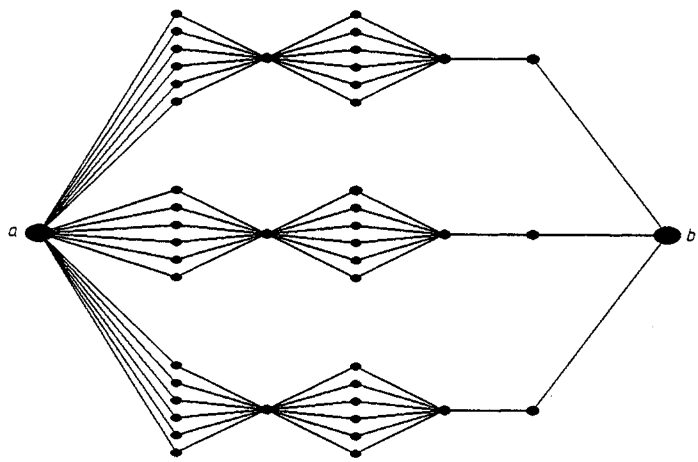

Figure 2. The element $S_{l,m,n}$ of the set of possible selectors for $p=\frac{3}{4}$. In this example, for $i=1$ and $j=3$, we have $l=3$, $m=6$ and $n=1$.

and, using only the first term of $g_{m,n}$,

$$P_p[H_B(S_{l,m,n})] = h_l\{p^{nj}(1-p)^{ni}[1-(1-p)^m]^{n(j-i)}\}.$$

Our construction was motivated by the fact that $g(p) \equiv p^j(1-p)^i$ attains its unique maximum at $j/(i+j) = p$ and is monotonic on each side of p. We will see later that the relevant behaviour of $g_{n,m}(x)$ is essentially the same as that of $g^n(x)$, so we begin by choosing n and l to satisfy conditions involving $g(x)$.

We now wish to determine n and l such that both

$$1-[1-g(x)^n - g(1-x)^n]^l < \varepsilon \tag{$*$}$$

for $x \notin [p-\delta, p+\delta]$ and

$$1-[1-g(p)^n]^l > 1-\varepsilon. \tag{$**$}$$

For inequality $(*)$, it is sufficient to show that

$$l \log[1-2g(x_0)^n] > \log(1-\varepsilon)$$

where x_0 denotes either $p-\delta$ or $p+\delta$, whichever provides the largest value of g. Using $1-x<\mathrm{e}^{-x}$, this is satisfied if

$$l < -[\log(1-\varepsilon)]/2g(x_0)^n.$$

Inequality $(**)$ is equivalent to

$$l \log[1-g(p)^n] < \log \varepsilon$$

which, using $1-x > \mathrm{e}^{-2x}$ for $x>0$ sufficiently small, is satisfied if

$$l > -[\log(\varepsilon)]/2g(p)^n.$$

Thus it is possible to choose l to satisfy both $(*)$ and $(**)$ if

$$\log(\varepsilon)/\log(1-\varepsilon) < g(p)^n/g(x_0)^n. \tag{$***$}$$

Since $g(p) > g(x_0)$, the ratio on the left can be made arbitrarily large by choosing n sufficiently large. Thus, choose n so that $(***)$ holds and then l to satisfy both $(*)$ and $(**)$.

We now return to the exact probability of passage through the selector. Note that, for any m and any $x \notin [p-\delta, p+\delta]$,

$$g_{n,m}(x) < g(x)^n + g(1-x)^n \leqslant 2g(x_0)^n$$

and that $h_l(x)$ is monotonic increasing. Thus, with the previous choices of n and l, we have

$$P_x[H(S_{l,m,n})] < h_l(2g(x_0)^n) \leqslant \varepsilon$$

for all $x \notin [p-\delta, p+\delta]$, independently of the choice of m.

Note also that $g_{n,m}(p) \geqslant g(p)^n[1-(1-p)^m]^{n(j-i)}$. Since $h_l(x)$ is continuous and $h_l(g(p)^n) > 1-\varepsilon$, we may choose m sufficiently large that $h_l\{g(p)^n[1-(1-p)^m]^{n(j-i)}\} > 1-\varepsilon$. Then by monotonicity of h_l,

$$P_p[H_B(S_{l,m,n})] \geqslant h_l\{g(p)^n[1-(1-p)^m]^{n(j-i)}\} > 1-\varepsilon.$$

Theorem. For any positive integer N, let $I_1, I_2, \ldots, I_N$ be a collection of disjoint open subintervals of $(\frac{1}{2}, 1)$. Let $p_i \in I_i$, $1 \leqslant i \leqslant N$. There exists an infinite connected graph G such that $\theta_v^{AB}(p_i, G) > 0$ for all $i = 1, 2, \ldots, N$ and $\theta_v^{AB}(p, G) = 0$ for $p \in (\frac{1}{2}, 1) \cap (\bigcup I_i)^c$.

Proof. Since each I_i is an open interval, for each p_i we may choose δ_i so that the interval $(p_i - \delta_i, p_i + \delta_i) \subset I_i$. Let $p^* = \min\{p_i: 1 \leqslant i \leqslant N\}$, and choose ε such that $p^*(1-\varepsilon)^6 > \frac{1}{2}$ and $N\varepsilon < \sin(\pi/18)$. Replace each edge of the triangular lattice T by a graph obtained by identifying the initial vertices and the final vertices of two copies of each $(p_i, \delta_i, \varepsilon)$ selector, $i = 1, 2, 3, \ldots, N$. This provides a candidate graph, which we now show satisfies the conditions of the theorem.

To prove non-existence of AB percolation outside of the intervals I_i, we introduce an associated bond percolation model. Consider an edge of the triangular lattice to be open if there is an AB path through any of the selectors used to replace that edge. For $x \in (\bigcup I_i)^c$, x is not in any of the intervals $(p_i - \delta, p_i + \delta)$, so the probability of an AB path through each specific selector is less than ε. Since there are $2N$ selectors, the probability that each edge in T is open is at most $2N\varepsilon$. By assumption, $2N\varepsilon < 2\sin(\pi/18)$, which is the critical probability of bond percolation on the triangular lattice. Thus, AB percolation is impossible on the constructed graph.

To show existence of AB percolation at a specific p_i, associate one of the p_i selectors replacing a given edge with each of the endpoints. Introduce an associated classical site percolation model on the triangular lattice by declaring a vertex of T to be open if it is labelled A and all of its p_i selectors contain AB paths which start at a neighbour labelled B (i.e. H_B occurs for each selector). Note that if there is an infinite open path in T, then there is an infinite AB path in the constructed graph. Since $p_i(1-\varepsilon)^6 \geqslant p^*(1-\varepsilon)^6 > \frac{1}{2}$, which is the site percolation critical probability for the triangular lattice, AB percolation occurs at p_i. (Note that we may ignore the other selectors in the existence proof, since they only increase the likelihood of an infinite AB cluster.)

If p is irrational, we may choose a rational number sufficiently close to p that, for the corresponding i and j, $g(p)$ is strictly greater than the maximum of $g(p-\delta)$ and $g(p+\delta)$, then may apply the argument above.

Note that all of the selectors are planar, so all the graphs constructed in our proof are planar graphs. To include $\frac{1}{2}$ in the set $\{p_i\}$, $(\frac{1}{2}, \delta, \varepsilon)$ selectors may be constructed by the same procedure, as noted above. However, to obtain AB percolation in a neighbourhood of $\frac{1}{2}$, one must start with an underlying graph with site percolation critical probability strictly less than $\frac{1}{2}$. If such a graph is periodic and has one axis of symmetry, then it cannot be planar. However, Wierman (1984) has constructed fully triangulated planar graphs with site percolation critical probability strictly less than $\frac{1}{2}$, so again a planar example may be constructed.

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